NON-COMMUTATIVE L_p-SPACES ASSOCIATED WITH A MAHARAM TRACE

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Abstract. Non-commutative L_p-spaces $L^p(M, \Phi)$ associated with the Maharam trace are defined and their dual spaces are described.

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1. Introduction

Development of the theory of integration for measures μ with the values in Dedekind complete Riesz spaces has inspired the study of (bo)-complete lattice-normed spaces $L^p(\mu)$ (see, for example, [1], 6.1.8). Note that, if the measure μ satisfies the Maharam property, then the spaces $L^p(\mu)$ are Banach-Kantorovich.

The existence of center-valued traces on finite von Neumann algebras naturally leads to a study of the integration for traces with the values in a complex Dedekind complete Riesz space $F_{\mathbb{C}} = F \oplus iF$. For commutative von Neumann algebras, the development of $F_{\mathbb{C}}$-valued integration is a part of the study of the properties of order continuous positive maps of Riesz spaces, for which we refer to the treatise by A.G. Kusraev [1]. The operators possessing the Maharam property provide important examples of such mappings, while the L^p-spaces associated with such operators are non-trivial examples of Banach-Kantorovich Riesz spaces.

Let M be a non-commutative von Neumann algebra, $F_{\mathbb{C}}$ a von Neumann subalgebra in the center of M, and let $\Phi : M \to F_{\mathbb{C}}$ be a trace such that $\Phi(zx) = z\Phi(x)$ for all $z \in F_{\mathbb{C}},\ x \in M$. Then the non-commutative L^p-space $L^p(M, \Phi)$ is a Banach-Kantorovich space [2], [3], and the trace Φ satisfies the Maharam property, that is, if $0 \leq z \leq \Phi(x),\ z \in F_{\mathbb{C}},\ 0 \leq x \in M$, then there exists $0 \leq y \leq x$ such that $\Phi(y) = z$ (compare with [1], 3.4.1).

In [4], a faithful normal trace Φ on M with the values in an arbitrary complex Dedekind complete Riesz space was considered. In particular, a complete description of such traces in the case when Φ is a Maharam trace was given. In the same paper, utilizing the locally measure topology on the algebra $S(M)$ of all measurable operators affiliated with M, the Banach-Kantorovich space

$L^1(M,\Phi) \subset S(M)$ was constructed and a version of Radon-Nikodym-type theorem for Maharam traces was established.

In the present article, we define a new class of Banach-Kantorovich spaces, non-commutative L_p-spaces $L^p(M,\Phi)$ associated with a Maharam trace; also, we give a description of their dual spaces. We use the terminology and results of the theory of von Neumann algebras ([5], [6]), the theory of measurable operators ([7], [8]), and of the theory of Dedekind complete Riesz space and Banach-Kantorovich spaces ([1]).

2. Preliminaries

Let X be a vector space over the field $\mathbb{C}$ of complex numbers, and let F be a Riesz space. A mapping $\|\cdot\| : X \to F$ is said to be a vector (F-valued) norm if it satisfies the following axioms:

(1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$ $(x \in X)$;
(2) $\|\lambda x\| = |\lambda|\, \|x\|$ $(\lambda \in \mathbb{C},\, x \in X)$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in X)$.

A norm $\|\cdot\|$ is called decomposable if the following property holds:

Property 1. *If $f_1, f_2 \geq 0$ and $\|x\| = f_1 + f_2$, then there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\|x_k\| = f_k$ $(k = 1, 2)$.*

If property 1 is valid only for disjoint elements $f_1, f_2 \in F$, the norm is called disjointly decomposable or, briefly, d-decomposable.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space (shortly, LNS). If the norm $\|\cdot\|$ is decomposable (d-decomposable), then so is the space $(X, \|\cdot\|)$.

A net $\{x_\alpha\}_{\alpha \in A} \subset X$ (bo)- converges to $x \in X$ if the net $\{\|x_\alpha - x\|\}_{\alpha \in A}$ (o)-converges to zero in the Riesz space F. A net $\{x_\alpha\}_{\alpha \in A}$ is said to be a (bo)-Cauchy net if $\sup\limits_{\alpha,\beta \geq \gamma} \|x_\alpha - x_\beta\| \downarrow 0$. An LNS is called (bo)- complete if any (bo)-Cauchy net (bo)-converges. A Banach-Kantorovich space (shortly, BKS) is a d-decomposable (bo)-complete LNS. It is well known that every BKS is a decomposable LNS.

Let F be a Dedekind complete Riesz space, and let $F_{\mathbb{C}} = F \oplus iF$ be the complexification of F. If $z = \alpha + i\beta \in F_{\mathbb{C}}$, $\alpha, \beta \in F$, then $\overline{z} := \alpha - i\beta$, and $|z| := \sup\{Re(e^{i\theta}z) : 0 \leq \theta < 2\pi\}$ (see[1], 1.3.13).

Let $(X, \|\cdot\|_X)$ be the BKS over F. A linear operator $T : X \to F_{\mathbb{C}}$ is said to be F-bounded if there exists $0 \leq c \in F$ such that $|T(x)| \leq c\|x\|_X$ for all $x \in X$. For any F-bounded operator T, define the element $\|T\| = \sup\{|T(x)| : x \in X,$ $\|x\|_X \leq \mathbf{1}_F\}$, which is called the abstract F-norm of the operator T ([1], 4.1.3). It is known that $|T(x)| \leq \|T\|\, \|x\|_X$ for all $x \in X$ ([1], 4.1.1).

The set X^* of all F-bounded linear mappings from X into $F_{\mathbb{C}}$ is called the F-dual space to the BKS X. For $T, S \in X^*$, we set $(T+S)(x) = Tx + Sx$, $(\lambda T)(x) = \lambda T x$, where $x \in X,\ \lambda \in \mathbb{C}$. It is clear that X^* is a linear space with respect to the introduced algebraic operations. Moreover, $(X^*, \|\cdot\|)$ is a BKS ([1], 4.2.6).

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let $\mathbf{1}$ be the identity operator on H. Given a von Neumann algebra M acting on H, denote by $Z(M)$ the center of M and by $P(M)$ the lattice of all projections in M. Let $P_{fin}(M)$ be the set of all finite projections in M.

A densely-defined closed linear operator x (possibly unbounded) affiliated with M is said to be *measurable* if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow \mathbf{1}$, $\ p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = \mathbf{1} - p_n \in P_{fin}(M)$ for every $n = 1, 2, \dots$ (here $\mathfrak{D}(x)$ is the domain of x). Let us denote by $S(M)$ the set of all measurable operators.

Let x, y be measurable operators. Then $x + y,\ xy$ and x^* are densely-defined and preclosed. Moreover, the closures $\overline{x+y}$ (strong sum), $\overline{xy}$ (strong product) and x^* are also measurable, and $S(M)$ is a $*$-algebra with respect to the strong sum, strong product, and the adjoint operation (see [7]). For any subset $E \subset S(M)$ we denote by E_h (resp. E_+) the set of all self-adjoint (resp. positive) operators from E.

For $x \in S(M)$ let $x = u|x|$ be the polar decomposition, where $|x| = (x^*x)^{\frac{1}{2}}$, u is a partial isometry in $B(H)$. Then $u \in M$ and $|x| \in S(M)$. If $x \in S_h(M)$ and $\{E_\lambda(x)\}$ are the spectral projections of x, then $\{E_\lambda(x)\} \subset P(M)$.

Let M be a commutative von Neumann algebra. Then M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded complex measurable functions with the identification almost everywhere, where (Ω, Σ, μ) is a measurable space. In addition $S(M) \cong L^0(\Omega, \Sigma, \mu)$, where $L^0(\Omega, \Sigma, \mu)$ is the $*$-algebra of all complex measurable functions with the identification almost everywhere [7].

The locally measure topology $t(M)$ on $L^0(\Omega, \Sigma, \mu)$ is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods of 0 is given by

$$W(B, \varepsilon, \delta) = \{f \in\ L^0(\Omega,\, \Sigma,\, \mu):\ \text{there exists a set}\ \ E \in \Sigma,\ \text{such that}$$

$$E \subseteq B,\ \mu(B \setminus E) \leqslant \delta,\ f\chi_E \in L^\infty(\Omega, \Sigma, \mu),\ \|f\chi_E\|_{L_\infty(\Omega,\Sigma,\mu)} \leqslant \varepsilon\}.$$

Here $\varepsilon,\ \delta$ run over all strictly positive numbers and $B \in \Sigma,\ \ \mu(B) < \infty$. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra.

It is clear that zero neighborhoods $W(B, \varepsilon, \delta)$ are closed and have the following property: if $f \in W(B, \varepsilon, \delta)$, $g \in L^\infty(\Omega, \Sigma, \mu)$, $\|g\|_{L\infty(\Omega,\Sigma,\mu)} \le 1$, then $gf \in W(B, \varepsilon, \delta)$.

A net $\{f_\alpha\}$ converges locally in measure to f (notation: $f_\alpha \xrightarrow{t(M)} f$) if and only if $f_\alpha\chi_B$ converges in μ-measure to $f\chi_B$ for each $B \in \Sigma$ with $\mu(B) < \infty$. If M is σ-finite then there exists a faithful finite normal trace τ on M. In this case, the topology $t(M)$ is metrizable, and convergence $f_n \xrightarrow{t(M)} f$ is equivalent to convergence in trace τ of the sequence f_n to f.

Let now M be an arbitrary finite von Neumann algebra, $\Phi_M : M \to Z(M)$ be a center-valued trace on M ([5], 7.11). Let $Z(M) \cong L^\infty(\Omega, \Sigma, \mu)$. The locally measure topology $t(M)$ on $S(M)$ is the linear (Hausdorff) topology whose fundamental system of neighborhoods of 0 is given by

$$V(B, \varepsilon, \delta) = \{x \in S(M)\colon \text{ there exists } p \in P(M), z \in P(Z(M))$$

$$\text{such that } xp \in M, \|xp\|_M \leqslant \varepsilon,\ z^\perp \in W(B, \varepsilon, \delta),\ \Phi_M(zp^\perp) \leqslant \varepsilon z\},$$

where $\|\cdot\|_M$ is the C^*-norm in M. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra [9].

From ([8], §3.5) we have the following criterion for convergence in the topology $t(M)$.

Proposition 2.1. *A net $\{x_\alpha\}_{\alpha\in A} \subset S(M)$ converges to zero in the topology $t(M)$ if and only if $\Phi_M(E^\perp_\lambda(|x_\alpha|) \xrightarrow{t(M)} 0$ for any $\lambda > 0$.*

Let M be an arbitrary von Neumann algebra, and let F be a Dedekind complete Riesz space. An *$F_{\mathbb{C}}$-valued trace* on the von Neumann algebra M is a linear mapping $\Phi : M \to F_{\mathbb{C}}$ with $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_h) \subset F,\ \Phi(M_+) \subset F_+ = \{a \in F : a \geq 0\}$. A trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$, *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_h,\ x_\alpha \uparrow x$.

If M is a finite von Neumann algebra, then its canonical center-valued trace $\Phi_M : M \to Z(M)$ is an example of a $Z(M)$-valued faithful normal trace.

Let us list some properties of the trace $\Phi : M \to F_{\mathbb{C}}$.

Proposition 2.2. *([4]) (i) Let $x, y, a, b \in M$. Then*
$\Phi(x^*) = \overline{\Phi(x)}$, $\Phi(xy) = \Phi(yx)$, $\Phi(|x^*|) = \Phi(|x|)$,
$|\Phi(axb)| \leq \|a\|_M\|b\|_M\Phi(|x|)$;
(ii) If Φ is a faithful trace, then M is finite;
(iii) If $x_n, x \in M$ and $\|x_n - x\|_M \to 0$, then $|\Phi(x_n) - \Phi(x)|$ relative uniform converges to zero;
(iv) $\Phi(|x + y|) \leq \Phi(|x|) + \Phi(|y|)$ for all $x, y \in M$.

The trace $\Phi : M \to F_{\mathbb{C}}$ possesses *the Maharam property* if for any $x \in M_+,\ 0 \leq f \leq \Phi(x),\ f \in F$, there exists $y \in M_+,\ y \leq x$ such that $\Phi(y) = f$. A faithful normal $F_{\mathbb{C}}$-valued trace Φ with the Maharam property is called *a*

Maharam trace (compare with [1], III, 3.4.1). Obviously, any faithful finite numerical trace on M is a $\mathbb{C}$-valued Maharam trace.

Let us give another examples of Maharam traces. Let M be a finite von Neumann algebra, let $\mathscr{A}$ be a von Neumann subalgebra in $Z(M)$, and let $T : Z(M) \to \mathscr{A}$ be an injective linear positive normal operator. If $f \in S(\mathscr{A})$ is a reversible positive element, then $\Phi(T,f)(x) = fT(\Phi_M(x))$ is an $S(\mathscr{A})$-valued faithful normal trace on M. In addition, if $T(ab) = aT(b)$ for all $a \in \mathscr{A}, b \in Z(M)$, then $\Phi(T,f)$ is a Maharam trace on M.

If τ is a faithful normal finite numerical trace on M and $\dim(Z(M)) > 1$, then $\Phi(x) = \tau(x)\mathbf{1}$ is a $Z(M)$-valued faithful normal trace, which does not possess the Maharam property (see [4]).

Let F have a weak order unit $\mathbf{1}_F$. Denote by $B(F)$ the complete Boolean algebra of unitary elements with respect to $\mathbf{1}_F$, and let Q be the Stone compact space of the Boolean algebra $B(F)$. Let $C_\infty(Q)$ be the Dedekind complete Riesz space of all continuous functions $a : Q \to [-\infty, +\infty]$ such that $a^{-1}(\{\pm\infty\})$ is a nowhere dense subset of Q. We identify F with the order-dense ideal in $C_\infty(Q)$ containing algebra $C(Q)$ of all continuous real functions on Q. In addition, $\mathbf{1}_F$ is identified with the function equal to 1 identically on Q ([1], 1.4.4).

We need the following theorem from [4].

Theorem 2.3. *Let Φ be an $F_\mathbb{C}$-valued Maharam trace on a von Neumann algebra M. Then there exists a von Neumann subalgebra $\mathscr{A}$ in $Z(M)$, a $*$-isomorphism ψ from $\mathscr{A}$ onto the $*$-algebra $C(Q)_\mathbb{C}$, a positive linear normal operator $\mathscr{E}$ from $Z(M)$ onto $\mathscr{A}$ with $\mathscr{E}(\mathbf{1}) = \mathbf{1}$, $\mathscr{E}^2 = \mathscr{E}$, such that*
1) $\Phi(x) = \Phi(\mathbf{1})\psi(\mathscr{E}(\Phi_M(x)))$ *for all* $x \in M$;
2) $\Phi(zy) = \Phi(z\mathscr{E}(y))$ *for all* $z, y \in Z(M)$;
3) $\Phi(zy) = \psi(z)\Phi(y)$ *for all* $z \in \mathscr{A}$, $y \in M$.

Due to Theorem 2.3, the $*$-algebra $\mathscr{B} = C(Q)_\mathbb{C}$ is a commutative von Neumann algebra, and $*$-algebra $C_\infty(Q)_\mathbb{C}$ is identified with the $*$-algebra $S(\mathscr{B})$. It is clear that the $*$-isomorphism ψ from $\mathscr{A}$ onto $\mathscr{B}$ can be extended to a $*$-isomorphism from $S(\mathscr{A})$ onto $S(\mathscr{B})$. We denote this mapping also by ψ.

Let Φ be a $S(\mathscr{B})$-valued Maharam trace on a von Neumann algebra M. A net $\{x_\alpha\} \subset S(M)$ converges to $x \in S(M)$ with respect to the trace Φ (notation: $x_\alpha \xrightarrow{\Phi} x$) if $\Phi(E_\lambda^\perp(|x_\alpha - x|)) \xrightarrow{t(\mathscr{B})} 0$ for all $\lambda > 0$.

Proposition 2.4. *([4]) $x_\alpha \xrightarrow{\Phi} x$ iff $x_\alpha \xrightarrow{t(M)} x$.*

An operator $x \in S(M)$ is said to be Φ-*integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \stackrel{\Phi}{\to} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathscr{B})} 0$ as $n, m \to \infty$.

Let x be a Φ-integrable operator from $S(M)$. Then there exists a $\widehat{\Phi}(x) \in S(\mathscr{B})$ such that $\Phi(x_n) \xrightarrow{t(\mathscr{B})} \widehat{\Phi}(x)$. In addition $\widehat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{\Phi} x$, $\Phi(|x_n - x_m|) \xrightarrow{t(\mathscr{B})} 0$ [4]. It is clear that each operator $x \in M$ is Φ-integrable and $\widehat{\Phi}(x) = \Phi(x)$.

Denote by $L^1(M, \Phi)$ the set of all Φ-integrable operators from $S(M)$. If $x \in S(M)$ then $x \in L^1(M, \Phi)$ iff $|x| \in L^1(M, \Phi)$, in addition $|\widehat{\Phi}(x)| \leq \widehat{\Phi}(|x|)$ [4]. For any $x \in L^1(M, \Phi)$, set $\|x\|_{1,\Phi} = \widehat{\Phi}(|x|)$. It is known that $L^1(M, \Phi)$ is a linear subspace of $S(M)$, $ML^1(M, \Phi)M \subset L^1(M, \Phi)$, and $x^* \in L^1(M, \Phi)$ for all $x \in L^1(M, \Phi)$ [4]. Moreover, the following theorem is true.

Theorem 2.5. *([4]) (i)* $(L^1(M, \Phi), \|\cdot\|_{1,\Phi})$ *is a Banach-Kantorovich space;*

(ii) $S(\mathscr{A})L^1(M, \Phi) \subset L^1(M, \Phi)$, *in addition* $\widehat{\Phi}(zx) = \psi(z)\widehat{\Phi}(x)$ *for all* $z \in S(\mathscr{A})$, $x \in L^1(M, \Phi)$.

3. L_p-spaces associated with a Maharam trace

Let $\mathscr{B}$ be a commutative von Neumann algebra, which is $*$-isomorphic to a von Neumann subalgebra $\mathscr{A}$ in $Z(M)$, and let $\Phi : M \to S(\mathscr{B})$ be a Maharam trace on M (see Theorem 2.3). For any $p > 1$, set $L^p(M, \Phi) = \{x \in S(M) : |x|^p \in L^1(M, \Phi)\}$ and $\|x\|_{p,\Phi} = \widehat{\Phi}(|x|^p)^{\frac{1}{p}}$. It is clear that $M \subset L^p(M, \Phi)$.

Let e be a nonzero projection in $\mathscr{B}$, and put $\Phi_e(a) = \Phi(a)e$, $a \in M$. A mapping $\Phi_e : M \to S(\mathscr{B}e)$ is a normal (not necessarily faithful) $S(\mathscr{B}e)$-valued trace on M. Denote by $s(\Phi_e) := \mathbf{1} - \sup\{p \in P(M) : \Phi_e(p) = 0\}$ the support of the trace Φ_e. It is clear that $s(\Phi_e) \in P(Z(M))$ and $\Phi_e(a) = \Phi(as(\Phi_e))$ is a faithful normal $S(\mathscr{B}e)$-valued trace on $Ms(\Phi_e)$ (compare [5], 5.15). Moreover Φ_e possesses the Maharam property.

If e and g are orthogonal nonzero projections in $P(\mathscr{B})$, then $\Phi_g(s(\Phi_e)) = \Phi(s(\Phi_e))g = \Phi_e(\mathbf{1})g = \Phi(\mathbf{1})eg = 0$, i.e. $s(\Phi_e)s(\Phi_g) = 0$. Let $\{e_i\}_{i\in I}$ be a family of nonzero mutually orthogonal projections in $P(\mathscr{B})$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathscr{B}}$, where $\mathbf{1}_{\mathscr{B}}$ is the unit of the algebra $\mathscr{B}$. If $z = \mathbf{1} - \sup\limits_{i\in I} s(\Phi_{e_i})$ then $\Phi(z)e_i = \Phi_{e_i}(z) = 0$ for all $i \in I$. Therefore $\Phi(z) = 0$, i.e. $z = 0$, or $\sup\limits_{i\in I} s(\Phi_{e_i}) = \mathbf{1}$.

Further, we need the following

Proposition 3.1. *Let* $x \in S(M)$ *and let* $\{e_i\}_{i\in I}$ *be the family of nonzero mutually orthogonal projections in* $P(\mathscr{B})$ *with* $\sup_{i\in I} e_i = \mathbf{1}_{\mathscr{B}}$. *Then* $x \in L^p(M, \Phi)$ *if and only if* $xs(\Phi_e) \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ *for all* $i \in I$. *In addition* $\|x\|_{p,\Phi}e_i = \|xs(\Phi_{e_i})\|_{p,\Phi_{e_i}}$.

Proof. Let $x \in L^p(M, \Phi)$, $a_n = E_n(|x|^p)|x|^p$ where $E_n(|x|^p)$ is the spectral projection of $|x|^p$ corresponding to the interval $(-\infty, n]$. It is clear that $a_n \xrightarrow{\Phi} |x|^p$

and $\Phi(|a_n - a_m|) \xrightarrow{t(\mathscr{B})} 0$ as $n, m \to \infty$. Hence, $a_n s(\Phi_{e_i}) \xrightarrow{\Phi_{e_i}} |x|^p s(\Phi_{e_i})$ (see Proposition 2.4). In addition, from the inequality $\Phi_{e_i}(|a_n s(\Phi_{e_i}) - a_m s(\Phi_{e_i})|) = \Phi(|a_n - a_m| s(\Phi_{e_i})) \le \Phi(|a_n - a_m|)$, we have $\Phi_{e_i}(|a_n s(\Phi_{e_i}) - a_m s(\Phi_{e_i})| \xrightarrow{t(\mathscr{B}e_i)} 0$. This means that $|xs(\Phi_{e_i})|^p = |x|^p s(\Phi_{e_i}) \in L^1(Ms(\Phi_{e_i}), \Phi_{e_i})$ and $\|xs(\Phi_{e_i})\|_{p,\Phi_{e_i}} = \widehat{\Phi}_{e_i}(|x|^p s(\Phi_{e_i}))^{\frac{1}{p}} = (\widehat{\Phi}(|x|^p)e_i)^{\frac{1}{p}} = \|x\|_{p,\Phi} e_i$.

Conversely, let $xs(\Phi_{e_i}) \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ for all $i \in I$. Set $a_{n,i} = E_n(|xs(\Phi_{e_i})|^p)|xs(\Phi_{e_i})|^p$. It is clear that $a_{n,i} \uparrow |xs(\Phi_{e_i})|^p = |x|^p s(\Phi_{e_i})$ as $n \to \infty$ for any fixed $i \in I$. Therefore $a_{n,i} \xrightarrow{t(Ms(\Phi_{e_i}))} |x|^p s(\Phi_{e_i})$, $\Phi_{e_i}(|a_{n,i} - a_{m,i}|) \xrightarrow{t(\mathscr{B}e_i)} 0$ as $n, m \to \infty$. Since $0 \le \Phi(\sqrt{a_{n,i}} a_{m,j} \sqrt{a_{n,i}}) = \Phi(a_{n,i} a_{m,j}) \le \|a_{m,j}\|_M \Phi(a_{n,i}) = \|a_{m,j}\|_M \Phi(a_{n,i}) e_i$ and $\Phi(a_{n,i} a_{m,j}) \le \|a_{n,i}\|_M \Phi(a_{m,j}) e_j$, we have $\Phi(a_{n,i} a_{m,j}) = 0$. Hence, $a_{n,i} a_{m,j} = 0$ for all $n, m,\ i \ne j$. Since $0 \le a_{n,i} \le ns(\Phi_{e_i})$, $s(\Phi_{e_i}) s(\Phi_{e_j}) = 0$, $i \ne j$, there is an $x_n \in M_+$ such that $x_n s(\Phi_{e_i}) = a_{n,i}$. Using the equality $\sup\limits_{i \in I} s(\Phi_{e_i}) = \mathbf{1}$, we obtain $x_n \xrightarrow{t(M)} |x|^p$ ([10]), moreover $\Phi(|x_n - x_m|) \xrightarrow{t(\mathscr{B})} 0$. Therefore $x \in L^p(M, \Phi)$. $\square$

Similar to in the case of the space $L^1(M, \Phi)$, the subset $L^p(M, \Phi)$ is invariant with respect to the action of involution in $S(M)$. The following proposition is devoted to this fact.

Proposition 3.2. *If $x \in L^p(M, \Phi)$, then $x^* \in L^p(M, \Phi)$ and $\|x\|_{p,\Phi} = \|x^*\|_{p,\Phi}$.*

Proof. Let $x = u|x|$ be the polar decomposition of x. Since an algebra M has a finite type, we can suppose that u is a unitary operator in M. For each $y \in S(M)$, we set $U(y) = uyu^*$. Then the mapping $U : S(M) \to S(M)$ is a $*$-isomorphism, and therefore $U(\varphi(y)) = \varphi(U(y))$ for any continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ and $y \in S_+(M)$ [10]. If $\varphi(t) = t^p$, $p > 1$, $t \ge 0$, and $y \in S_+(M)$ then $uy^p u^* = (uyu^*)^p$. In particular, we obtain the equality $|x^*|^p = u|x|^p u^*$. Hence, $x^* \in L^p(M, \Phi)$. Moreover $\|x^*\|_{p,\Phi} = \widehat{\Phi}(|x^*|^p)^{\frac{1}{p}} = \widehat{\Phi}(u|x|^p u^*)^{\frac{1}{p}} = \widehat{\Phi}(|x|^p)^{\frac{1}{p}} = \|x\|_{p,\Phi}$. $\square$

Now we need a version of the Hölder inequality for Maharam traces. In the proof of this inequality for numerical traces, properties of decreasing rearrangements of integrable operators are used [11]. For Maharam traces such theory of decreasing rearrangements does not exact. Therefore we use another approach connected with the concept of a bitrace on a C^*-algebra.

Let $\mathscr{N}$ be a C^*-algebra. A function $s : \mathscr{N} \times \mathscr{N} \to \mathbb{C}$ is called a bitrace on $\mathscr{N}$ ([12], 6.2.1) if the following relations hold:

(*i*) $s(x, y)$ is positively defined sesquilinear Hermitian form on $\mathscr{N}$;
(*ii*) $s(x, y) = s(x^*, y^*)$ for all $x, y \in \mathscr{N}$;
(*iii*) $s(zx, y) = s(x, z^* y)$ for all $x, y, z \in \mathscr{N}$;

(iv) for any $z \in \mathscr{N}$, the mapping $x \to zx$ is continuous on $(\mathscr{N}, \|\cdot\|_s)$ where $\|x\|_s = \sqrt{s(x,x)}$, $x \in \mathscr{N}$;

(v) the set $\{xy : x, y \in \mathscr{N}\}$ is dense in $(\mathscr{N}, \|\cdot\|_s)$.

If $\mathscr{N}$ has a unit, then condition (v) holds automatically.

Let us list examples of bitraces associated with the Maharam trace.

Let M be a von Neumann algebra, let $\Phi : M \to S(\mathscr{B})$ be a Maharam trace and let $Q = Q(P(\mathscr{B}))$ be the Stone compact space of the Boolean algebra $P(\mathscr{B})$. We claim that $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathscr{B}}$. If it is not the case, then $e = \mathbf{1}_{\mathscr{B}} - s(\Phi(\mathbf{1})) \neq 0$ and $z = \psi^{-1}(e) \neq 0$ where ψ is a $*$-isomorphism from Theorem 2.3. By Theorem 2.5(ii), we have $\Phi(z) = e\Phi(\mathbf{1}) = 0$, which contradicts to the faithfulness of the trace Φ. Thus, $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathscr{B}}$, and therefore the following elements are defined: $(\Phi(\mathbf{1}))^{-1} \in S_+(\mathscr{B})$ and $(\Phi(\mathbf{1}))^{-1}\Phi(x) \in C(Q)$ where $x \in M$. For any $t \in Q$, set $\varphi_t(x) = (\Phi(\mathbf{1})^{-1}\Phi(x))(t)$. It is clear that φ_t is a finite numerical trace on M. The function $s_t(x,y) = \varphi_t(y^*x) = \varphi_t(xy^*)$ is a bitrace on M. In fact, the conditions $(i) - (iii)$ are obvious. (iv) follows from the inequality $\|zx\|_{s_t} = \sqrt{\varphi_t((zx)^*(zx))} = \sqrt{\varphi_t(x^*z^*zx)} \leq \|z\|_M \|x\|_{s_t}$.

Let $s(x,y)$ be an arbitrary bitrace on a von Neumann algebra M. Set $N_s = \{x \in M : s(x,x) = 0\}$. It follows from ([12], 6.2.2) that N_s is a self-adjoint two-sided ideal in M. We consider the factor-space M/N_s with the scalar product $([x],[y])_s = s(x,y)$ where $[x],[y]$ are the equivalence classes from M/N_s with representatives x and y, respectively. Denote by $(H_s, (\cdot,\cdot)_s)$ the Hilbert space which is the completion of $(M/N_s, (\cdot,\cdot)_s)$. By the formula $\pi_s(x)([y]) = [xy]$, $x, y \in M$, one defines a $*$-homomorphism $\pi_s : M \to B(H_s)$. In addition $\pi_s(\mathbf{1}_M) = \mathbf{1}_{B(H_s)}$.

Denote by $U_s(M)$ the von Neumann subalgebra in $B(H_s)$ generated by operators $\pi_s(x)$, i.e. $U_s(M)$ is the closure of the $*$-subalgebra $\pi_s(M)$ in $B(H_s)$ with respect to the weak operator topology. According to ([13], s. 85-88), there exists a faithful normal semifinite numerical trace τ_s on $(U_s(M))_+$ such that $\tau_s(\pi(x^2)) = ([x],[x]) = s(x,x)$ for all $x \in M_+$. If φ is a trace on M and $s(x,y) = \varphi(y^*x)$ then $\tau_s(\pi_s(x^2)) = \varphi(x^2)$ for all $x \in M_+$. This means that $\tau_s(\pi_s(x)) = \varphi(x)$ for any $x \in M_+$. In addition, if $\varphi(\mathbf{1}_M) < \infty$, then $\tau_s(\mathbf{1}_{B(H_s)}) < \infty$. Consequently, τ_s is a faithful normal finite trace on $U_s(M)$.

Theorem 3.3. *Let Φ be a $S(\mathscr{B})$-valued Maharam trace on the von Neumann algebra M, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in L^p(M,\Phi)$, $y \in L^q(M,\Phi)$, then $xy \in L^1(M,\Phi)$ and $\|xy\|_{1,\Phi} \leq \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.*

Proof. We consider the bitrace $s_t(x,y) = \varphi_t(y^*x)$ on M where $\varphi_t(x) = ((\Phi(\mathbf{1}))^{-1}\Phi(x))(t)$, $t \in Q(P(\mathscr{B}))$. Denote by τ_t a faithful normal finite trace on $(U_{s_t}(M))_+$ such that $\tau_t(\pi_{s_t}(x)) = \varphi_t(x)$ for all $x \in M_+$. Since the trace

τ_t is finite, $\tau_t(\pi_{s_t}(x) = \varphi_t(x)$ for any $x \in M$. Let $L^p(U_{s_t}(M), \tau_t)$ be the non-commutative L^p-space associated with the numerical trace τ_t. It follows from [11] that

$$\|\pi_{s_t}(x)\pi_{s_t}(y)\|_{1,\tau_t} \leq \|\pi_{s_t}(x)\|_{p,\tau_t}\|\pi_{s_t}(y)\|_{q,\tau_t},$$

i.e.

$$\tau_t(|\pi_{s_t}(xy)|) \leq \tau_t(|\pi_{s_t}(x)|^p)^{\frac{1}{p}}\tau_t(|\pi_{s_t}(y)|^q)^{\frac{1}{q}}.$$

Since $\pi_{s_t}(|x|) = |\pi_{s_t}(x)|,\ x \in M$, we get $\pi_{s_t}(|x|^p) = (\pi_{s_t}(|x|))^p$ ([12], 1.5.3).

Thus,

$$\tau_t(\pi_{s_t}(|xy|)) \leq \tau_t(\pi_{s_t}(|x|^p))^{\frac{1}{p}}\tau_t(\pi_{s_t}(|y|^q))^{\frac{1}{q}},$$

i.e. $\varphi_t(|xy|) \leq \varphi_t(|x|^p)^{\frac{1}{p}}\varphi_t(|y|^q)^{\frac{1}{q}}$, or

$$(\Phi(\mathbf{1}))^{-1}\Phi(|xy|)(t) \leq [((\Phi(\mathbf{1}))^{-1}\Phi(|x|^p))(t)]^{\frac{1}{p}}[((\Phi(\mathbf{1}))^{-1}\Phi(|y|^q))(t)]^{\frac{1}{q}}$$

for all $t \in Q(P(\mathscr{B}))$. This means that

$$(\Phi(\mathbf{1}))^{-1}\Phi(|xy|) \leq [((\Phi(\mathbf{1}))^{-1}\Phi(|x|^p))]^{\frac{1}{p}}[((\Phi(\mathbf{1}))^{-1}\Phi(|y|^q))]^{\frac{1}{q}}.$$

Multiplying this inequality by $\Phi(\mathbf{1})$, we get $\|xy\|_{1,\Phi} \leq \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

Let now $x \in L^p_+(M,\Phi),\ y \in L^q_+(M,\Phi)$. We claim that $xy \in L^1(M,\Phi)$. Set $a_n = E_n(x)x,\ b_n = E_n(y)y$. We have $a_n, b_n \in M_+$ and $a_n \uparrow x,\ b_n \uparrow y$, in particular, $a_n \xrightarrow{\Phi} x,\ b_n \xrightarrow{\Phi} y$. Hence, $a_nb_n \in M$ and $a_nb_n \xrightarrow{\Phi} xy$. In addition, $\|a_nb_n - a_mb_m\|_{1,\Phi} \leq \|a_nb_n - a_nb_m\|_{1,\Phi} + \|a_nb_m - a_mb_m\|_{1,\Phi} \leq \|a_n\|_{p,\Phi}\|b_n - b_m\|_{q,\Phi} + \|a_n - a_m\|_{p,\Phi}\|b_m\|_{q,\Phi}$. Since $\|a_n\|_{p,\Phi} \leq \|x\|_{p,\Phi},\ \|b_m\|_{q,\Phi} \leq \|y\|_{q,\Phi}$, and for $n > m$, $\|a_n - a_m\|^p_{p,\Phi} = \widehat{\Phi}(x^pE_n(x)E^\perp_m(x)) \xrightarrow{t(\mathscr{B})} 0$, $\|b_n - b_m\|^q_{q,\Phi} = \widehat{\Phi}(y^qE_n(y)E^\perp_m(y)) \xrightarrow{t(\mathscr{B})} 0$, we get $\|a_nb_n - a_mb_m\|_{1,\Phi} \xrightarrow{t(\mathscr{B})} 0$ as $n, m \to \infty$. This means that $xy \in L^1(M,\Phi)$ and $\|a_nb_n - xy\|_{1,\Phi} \xrightarrow{t(\mathscr{B})} 0$. The inequality $|\|xy\|_{1,\Phi} - \|a_nb_n\|_{1,\Phi}| \leq \|xy - a_nb_n\|_{1,\Phi}$ implies $\|a_nb_n\|_{1,\Phi} \xrightarrow{t(\mathscr{B})} \|xy\|_{1,\Phi}$. Since

$$\|a_nb_n\|_{1,\Phi} \leq \|a_n\|_{p,\Phi}\|b_n\|_{q,\Phi} \xrightarrow{t(\mathscr{B})} \|x\|_{p,\Phi}\|y\|_{q,\Phi},$$

we obtain $\|xy\|_{1,\Phi} \leq \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

If $x \in L^p(M,\Phi)$ is arbitrary, $y \in L^q_+(M,\Phi)$ and $x = u|x|$ is the polar decomposition of x with the unitary $u \in M$, then $xy = u(|x|y) \in L^1(M,\Phi)$ and $\|xy\|_{1,\Phi} = \||x|y\|_{1,\Phi} \leq \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

Let now $x \in L^p(M,\Phi),\ y \in L^q(M,\Phi)$ be arbitrary and let $y^* = v|y^*|$ be the polar decomposition of y^* with the unitary $v \in M$. According to Proposition 3.2, $|y^*| \in L^q(M,\Phi)$ and $\|y^*\|_{q,\Phi} = \|y\|_{q,\Phi}$. Therefore $xy = (x|y^*|)v^* \in L^1(M,\Phi)$ and

$$\|xy\|_{1,\Phi} = \|x|y^*|\|_{1,\Phi} \leq \|x\|_{p,\Phi}\||y^*|\|_{q,\Phi} = \|x\|_{p,\Phi}\|y\|_{q,\Phi}.$$

$\square$

Theorem 3.4. *Let Φ, M, p, and q be the same as in Theorem 3.3. If $x \in S(M)$, $xy \in L^1(M,\Phi)$ for all $y \in L^q(M,\Phi)$ and the set $D(x) = \{|\widehat{\Phi}(xy)| : y \in L^q(M,\Phi),\ \|y\|_{q,\Phi} \leq \mathbf{1}_{\mathscr{B}}\}$ is bounded in $S_h(\mathscr{B})$, then $x \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} = \sup D(x)$.*

Proof. Let $x \neq 0$, and let $x = u|x|$ be the polar decomposition of x with the unitary $u \in M$. Set $y_n = |x|^{p-1}E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^*,\ n = 1, 2, \ldots$ It is clear that $y_n \in M$ and

$$xy_n = u|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^* = uE_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^* \geq 0.$$

On the other hand,

$$|y_n|^2 = uE_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)|x|^{2p-2}E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^* =$$

$$uE_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)|x|^{\frac{2p}{q}}E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^*,$$

and therefore $0 \leq |y_n|^q = (|y_n|^2)^{\frac{q}{2}} = xy_n$, in particular, $\|y_n\|_{q,\Phi} = \Phi(xy_n)^{\frac{1}{q}}$.

Since $xy_n \xrightarrow{t(M)} u|x|^p u^* \neq 0$, we have $xy_n \neq 0$ for all$n \geq n_0$. Set $e_n = s(\Phi(xy_n)$ as $n \geq n_0$. Since $S_h(\mathscr{B}) = C_\infty(Q(P(\mathscr{B}))$, there exists a unique $b_n \in S_+(\mathscr{B})e_n$ such that $b_n\Phi(xy_n) = e_n$. It is clear that $b_n^{\frac{1}{q}}\Phi^{\frac{1}{q}}(xy_n) = e_n$. If $z_n = \psi^{-1}(e_n)$, $a_n = \psi^{-1}(b_n^{\frac{1}{q}}) \in S(\mathscr{A}z_n)$, then by theorem 2.5$(ii)$, $a_n y_n \in L^q(M,\Phi)$ and $\|a_n y_n\|^q_{q,\Phi} = \widehat{\Phi}(a_n^q|y_n|^q) = b_n\widehat{\Phi}(xy_n) = e_n \leq \mathbf{1}_{\mathscr{B}}$. Hence, $|\widehat{\Phi}(a_n xy_n)| = |\widehat{\Phi}(x(a_n y_n))| \leq \sup D(x)$ for all $n \geq n_0$. On the other hand,

$$\widehat{\Phi}(a_n xy_n) = b_n^{\frac{1}{q}}\widehat{\Phi}(xy_n) = (b_n\widehat{\Phi}(xy_n))^{\frac{1}{q}}\widehat{\Phi}(xy_n)^{1-\frac{1}{q}} = \widehat{\Phi}(xy_n)^{\frac{1}{p}} =$$

$$\widehat{\Phi}(u|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)u^*)^{\frac{1}{p}} = \widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|))^{\frac{1}{p}}.$$

Since $(|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)) \uparrow |x|^p,\ |x|^p(E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|) \in M_+$ and

$\widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|)) \leq (\sup D(x))^p$, we have $|x|^p \in L^1(M,\Phi)$ and

$\widehat{\Phi}(|x|^p) = \sup_{n\geq 1}\widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{\frac{1}{n}}(|x|))$ [14]. This means that $x \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} \leq \sup D(x)$. Theorem 3.3 implies $\sup D(x) \leq \|x\|_{p,\Phi}$, and therefore $\|x\|_{p,\Phi} = \sup D(x)$. □

With the help of Theorem 3.4, it is not difficult to show that $L^p(M,\Phi)$ is disjointly decomposable LNS over $S_h(\mathscr{B})$ for all $p > 1$.

Theorem 3.5. (i) *$L^p(M,\Phi)$ is a linear subspace in $S(M)$, and $\|\cdot\|_{p,\Phi}$ is the disjointly decomposable $S_h(\mathscr{B})$-valued norm on $L^p(M,\Phi)$;*

(ii) *$ML^p(M,\Phi)M \subset L^p(M,\Phi)$, and $\|axb\|_{p,\Phi} \leq \|a\|_M\|b\|_M\|x\|_{p,\Phi}$ for all $a, b \in M,\ x \in L^p(M,\Phi)$;*

(iii) If $0 \leq x \leq y \in L^p(M,\Phi),\ x \in S(M)$, *then* $x \in L^p(M,\Phi)$ *and* $\|x\|_{p,\Phi} \leq \|y\|_{p,\Phi}$.

Proof. (i) It is clear that $\lambda x \in L^p(M,\Phi)$ and $\|\lambda x\|_{p,\Phi} = |\lambda|\|x\|_{p,\Phi}$ for all $x \in L^p(M,\Phi),\ \lambda \in \mathbb{C}$. Moreover, $\|x\|_{p,\Phi} \geq 0$ and $\widehat{\Phi}(|x|^p) = \|x\|^p_{p,\Phi} = 0$ if and only if $x = 0$.

We claim that $x + y \in L^p(M,\Phi)$ and $\|x+y\|_{p,\Phi} \leq \|x\|_{p,\Phi} + \|y\|_{p,\Phi}$ for each $x, y \in L^p(M,\Phi)$. By theorem 3.3, $(x+y)z = xz + yz \in L^1(M,\Phi)$ for all $z \in L^q(M,\Phi)$, in addition

$$|\widehat{\Phi}((x+y)z)| \leq |\widehat{\Phi}(xz)| + |\widehat{\Phi}(yz)|.$$

If $\|z\|_{q,\Phi} \leq \mathbf{1}_{\mathscr{B}}$, then by theorem 3.4,

$$|\widehat{\Phi}((x+y)z)| \leq \|x\|_{p,\Phi} + \|y\|_{p,\Phi}.$$

Using Theorem 3.4 again, we obtain $x+y \in L^p(M,\Phi)$ and $\|x+y\|_{p,\Phi} \leq \|x\|_{p,\Phi} + \|y\|_{p,\Phi}$. Thus, $L^p(M,\Phi)$ is a linear subspace in $S(M)$, and $\|\cdot\|_{p,\Phi}$ is a $S_h(\mathscr{B})$-valued norm on $L^p(M,\Phi)$.

Let us now show that the norm $\|\cdot\|_{p,\Phi}$ is d-decomposable. It is known [4] that, if $x \in L^1(M,\Phi),\ \|x\|_{1,\Phi} = f_1 + f_2$, where $f_1, f_2 \in S_+(\mathscr{B}),\ f_1 f_2 = 0$, then, setting $x_i = xp_i$ for $p_i = \psi^{-1}(s(f_i)),\ i = 1,2$, we get $x = x_1 + x_2$ and $\|x_i\|_\Phi = f_i,\ i = 1,2$.

Let $y \in L^p_+(M,\Phi),\ \|y\|_{p,\Phi} = g_1 + g_2$ where $g_1, g_2 \in S_+(\mathscr{B}),\ g_1 g_2 = 0$, i.e. $\|y^p\|_{1,\Phi} = \|y\|^p_{p,\Phi} = g_1^p + g_2^p$. Set $q_i = \psi^{-1}(s(g_i^p)) \in P(\mathscr{A}) \subset P(Z(M))$ and $y_i = yq_i$. Then $y_i^p = y^p q_i$ and using [4] for $x = y^p,\ f_i = g_i^p,\ i = 1,2$ we obtain that $y^p q_1 + y^p q_2 = y^p$ and $\|yq_i\|_{p,\Phi} = g_i,\ i = 1,2$. Since $q_1 q_2 = 0,\ q_1, q_2 \in P(Z(M))$, we have $yq_1 + yq_2 = y$.

Let now y be an arbitrary element from $L^p(M,\Phi)$ and let $y = u|y|$ be the polar decomposition of y with the unitary $u \in M$. Let $\||y|\|_{p,\Phi} = \|y\|_{p,\Phi} = f_1 + f_2$ where $f_1, f_2 \in S_+(\mathscr{B}),\ f_1 f_2 = 0$. It follows from above that for $q_i = \psi^{-1}(s(f_i^p)) \in P(\mathscr{A})$, we have $|y| = |y|q_1 + |y|q_2 \quad \||y|q_i\|_{p,\Phi} = f_i$. Consequently, $y = u|y| = u|y|q_1 + u|y|q_2 = yq_1 + yq_2$ and $\|yq_i\|_{p,\Phi} = \||yq_i|\|_{p,\Phi} = \||y|q_i\|_{p,\Phi} = f_i,\ i = 1,2$. Hence, the norm $\|\cdot\|_{p,\Phi}$ is d-decomposable.

(ii) Let v be a unitary operator in M, $x \in L^p(M,\Phi)$. Then $|vx| = (x^* v^* vx)^{\frac{1}{2}} = |x|$, and therefore $vx \in L^p(M,\Phi)$. Since any operator $a \in M$ is a linear combination of four unitary operators, we have $ax \in L^p(M,\Phi)$, due to (i).

We claim that $\|ax\|_{p,\Phi} \leq \|a\|_M \|x\|_{p,\Phi}$ for $a \in M,\ x \in L^p(M,\Phi)$. Let ν be a faithful normal semifinite numerical trace on $\mathscr{B}$. If for some $a \in M$, $x \in L^p(M,\Phi)$ the previous inequality is not true, then there are $\varepsilon > 0,\ 0 \neq e \in P(\mathscr{B}),\ \nu(e) < \infty$ such that

$$e\|ax\|_{p,\Phi} \geq e\|a\|_M \|x\|_{p,\Phi} + \varepsilon e.$$

By the formula

$$\tau(b) = \nu(e\Phi(b)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}), b \in Ms(\Phi_e)$$

one defines a faithful normal finite numerical trace on $Ms(\Phi_e)$. If $z = \psi^{-1}(e) \in P(\mathscr{A})$, then $\Phi_e(\mathbf{1}-z) = (\mathbf{1}_{\mathscr{B}}-e)e\Phi(\mathbf{1}) = 0$, i.e. $s(\Phi_e) \leq z$. Since $\Phi(z-s(\Phi_e)) = \Phi(z(\mathbf{1} - s(\Phi_e)) = e\Phi(\mathbf{1} - s(\Phi_e)) = 0$, we get $z = s(\Phi_e)$. We consider the L^p-space $L^p(Ms(\Phi_e), \tau)$ associated with the numerical trace τ, and let us show that $xz \in L^p(Ms(\Phi_e), \tau)$. Let $x_n = E_n(|x|)|x|$. It is clear that $0 \leq x_n^p z \uparrow |x|^p z$, moreover

$$\tau(x_n^p z) = \nu(e\Phi(x_n^p z)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) \leq \nu(e) < \infty.$$

Hence, $|xz|^p = |x|^p z \in L^p(Ms(\Phi_e), \tau)$ and $\|xz\|_{p,\tau}^p = \lim\limits_{n\to\infty} \|x_n^p z\|_{p,\tau}^p = \nu(e\widehat{\Phi}(|x|^p z)(\mathbf{1}_{\mathscr{B}}+\Phi(\mathbf{1})+\widehat{\Phi}(|x|^p))^{-1})$. Thus, if $a \in M$ then $axz \in L^p(Ms(\Phi_e), \tau)$, in addition

$$\|a\|_M \|xz\|_{p,\tau}^p \geq \|axz\|_{p,\tau}^p = \nu(e\widehat{\Phi}(|axz|^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) =$$

$$\nu(e\|ax\|_{p,\Phi}^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) \geq$$

$$\nu(e(\|a\|_M \|x\|_{p,\Phi} + \varepsilon)^p(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) > \|a\|_M^p \|xz\|_{p,\tau}^p,$$

which is not the case. Consequently, $\|ax\|_{p,\Phi} \leq \|a\|_M \|x\|_{p,\Phi}$.

If $b \in M$, $x \in L^p(M, \Phi)$, then by Proposition 3.2 and from above, we have $b^*x^* \in L^p(M, \Phi)$. Using Proposition 3.2 again, we obtain $xb = (b^*x^*)^* \in L^p(M, \Phi)$ and $\|xb\|_{p,\Phi} = \|b^*x^*\|_{p,\Phi} \leq \|b^*\|_M \|x^*\|_{p,\Phi} = \|b\|_M \|x\|_{p,\Phi}$.

(iii) Let $0 \leq x \leq y \in L^p(M, \Phi)$, $x \in S(M)$. It follows from ([8], §2.4) that $\sqrt{x} = a\sqrt{y}$ where $a \in M$ with $\|a\|_M \leq 1$. Hence, $x = \sqrt{x}(\sqrt{x})^* = aya^* \in L^p(M, \Phi)$ $\|x\|_{p,\Phi} \leq \|a\|_M \|a^*\|_M \|y\|_{p,\Phi} \leq \|y\|_{p,\Phi}$.

□

Using the Hölder inequality and the (bo)-completeness of the space $(L^1(M, \Phi), \|\cdot\|_\Phi)$ we can establish the (bo)-completeness of the space $(L^p(M, \Phi), \|\cdot\|_{p,\Phi})$.

Theorem 3.6. *Let Φ, M, p be the same as in Theorem 3.3. Then $(L^p(M, \Phi), \|\cdot\|_{p,\Phi})$ is the Banach-Kantorovich space.*

Proof. First, we assume that $\mathscr{B}$ is a σ-finite von Neumann algebra. Then there exists a faithful normal finite numerical trace ν on $\mathscr{B}$. The numerical function $\tau(a) = \nu(\Phi(a)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}))^{-1})$ is a faithful normal finite trace on M. Moreover, the topology $t(M)$ coincides with topology of convergence in measure t_τ in $(S(M), \tau)$ ([8], §3.5).

Let $\{x_\alpha\}_{\alpha \in A} \subset (L^p(M, \Phi), \|\cdot\|_{p,\Phi})$ be an (bo)-Cauchy net i.e. $b_\gamma = \sup\limits_{\alpha,\beta \geq \gamma} \|x_\alpha - x_\beta\|_{p,\Phi} \downarrow 0$. According to the Hölder inequality, for each $x \in L^p(M, \Phi)$ we have

$x \in L^1(M,\Phi)$ and $\|x\|_{1,\Phi} = \widehat{\Phi}(|x|\mathbf{1}) \leq (\Phi(\mathbf{1}))^{\frac{1}{q}}\|x\|_{p,\Phi}$. In particular, the set $\{\|x_\alpha - x_\beta\|_{1,\Phi}\}_{\alpha,\beta\geq\gamma}$ is bounded in $S_h(\mathscr{B})$, and $\sup\limits_{\alpha,\beta\geq\gamma}\|x_\alpha - x_\beta\|_{1,\Phi} \leq (\Phi(\mathbf{1}))^{\frac{1}{q}}b_\gamma$ for all $\gamma \in A$. Consequently [4], there exists $x \in L^1(M,\Phi)$ such that $\|x_\alpha - x\|_{1,\Phi} \xrightarrow{(o)} 0$ in particular, $x_\alpha \xrightarrow{t_\tau} x$ and $y_\alpha = |x_\alpha - x_\beta| \xrightarrow{t_\tau} |x - x_\beta|$. Since the function $\varphi(t) = t^p$ is continuous on $[0,\infty)$, the operator function $y \longmapsto y^p$ is continuous on $(S_+(M), t_\tau)$ [15]. Hence, $0 \leq y_\alpha^p \xrightarrow{t_\tau} |x - x_\beta|^p$, in addition $\widehat{\Phi}(y_\alpha^p) = \|x_\alpha - x_\beta\|_{p,\Phi}^p \leq b_\gamma^p$. Using the of Fatou's theorem [14], we obtain $|x - x_\beta|^p \in L^1(M,\Phi)$ and $\widehat{\Phi}(|x - x_\beta|^p) \leq b_\gamma^p$. Thus, $(x - x_\beta) \in L^p(M,\Phi)$ for all $\beta \geq \gamma$ and $\sup\limits_{\beta\geq\gamma}\|x - x_\beta\|_{p,\Phi} \leq b_\gamma \downarrow 0$. This means that $x \in L^p(M,\Phi)$, and $\|x_\alpha - x\|_{p,\Phi} \xrightarrow{(o)} 0$.

Now let $\mathscr{B}$ be an arbitrary von Neumann algebra (not necessarily σ-finite), and let $\{x_\alpha\} \subset L^p(M,\Phi)$ be a (bo)-Cauchy net. It follows from the above that there exists $x \in L^1(M,\Phi)$ such that $\|x_\alpha - x\|_{1,\Phi} \xrightarrow{(o)} 0$. In particular $x_\alpha \xrightarrow{t(M)} x$. Let ν be a faithful normal semifinite numerical trace on $\mathscr{B}$, and let $\{e_i\}_{i\in I}$ be the family of nonzero mutually orthogonal projections in $\mathscr{B}$ such that $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathscr{B}}$, and $\nu(e_i) < \infty$ for all $i \in I$. It is clear that $\{x_\alpha s(\Phi_{e_i})\}_{\alpha\in A}$ is a (bo)-Cauchy net in $L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$. Since the algebra $\mathscr{B}e_i$ is σ-finite, from the above there exists $x_i \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ such that $\|x_i - x_\alpha s(\Phi_{e_i})\|_{p,\Phi_{e_i}} \xrightarrow{(o)} 0$. In particular, $x_\alpha s(\Phi_{e_i}) \xrightarrow{t(M)} x_i = x_i s(\Phi_{e_i})$. On the other hand, convergence $x_\alpha \xrightarrow{t(M)} x$ implies $x_\alpha s(\Phi_{e_i}) \xrightarrow{t(M)} xs(\Phi_{e_i})$. Thus, $xs(\Phi_{e_i}) = x_i s(\Phi_{e_i})$ for all $i \in I$. By Proposition 3.1, we have $x \in L^p(M,\Phi)$ and $\|x - x_\alpha\|_{p,\Phi}e_i = \|xs(\Phi_{e_i}) - x_\alpha s(\Phi_{e_i})\|_{p,\Phi_{e_i}} \xrightarrow{(o)} 0$ for all $i \in I$ and therefore $\|x - x_\alpha\|_{p,\Phi} \xrightarrow{(o)} 0$. $\square$

Proposition 3.7. *If $\{x_\alpha\}_{\alpha\in A} \subset L_h^p(M,\Phi)$ and $x_\alpha \downarrow 0$, then $\|x_\alpha\|_{p,\Phi} \downarrow 0$.*

Proof. Let ν be a faithful normal semifinite numerical trace on $\mathscr{B}$. If $b = \inf_{\alpha\in I}\|x_\alpha\|_{p,\Phi} \neq 0$, then there are $\varepsilon > 0$, $0 \neq e \in P(\mathscr{B})$ with $\nu(e) < \infty$ such that $e\|x_\alpha\|_{p,\Phi} \geq eb \geq \varepsilon e$ for all $\alpha \in A$. Put $\Phi_e(x) = e\Phi(x)$, $x \in M$, and $\tau(y) = \nu(\Phi(y)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1})$, $y \in Ms(\Phi_e)$, where α_0 is a fixed element from A. Let us prove that $L^p(Ms(\Phi_e),\tau) \subset L^p(Ms(\Phi_e),\Phi_e)$ and $\|x\|_{p,\tau}^p = \nu(\widehat{\Phi}(|x|^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1})$ for all $x \in L^p(Ms(\Phi_e),\tau)$. It is sufficient to consider the case where $x \in L_+^p(Ms(\Phi_e),\tau)$. Set $x_n = E_n(x)xs(\Phi_e)$. It is clear that $x_n \in (Ms(\Phi_e))_+$, $x_n^p \uparrow x^p$, $x_n^p \xrightarrow{\tau} x^p$, and therefore $x_n^p \xrightarrow{t(M)} x^p$. Moreover, $\Phi(|x_n^p - x_m^p|) = \Phi(x^p E_n(x)E_m^\perp(x))$ as $m < n$. Since $\nu(e\Phi(|x_n^p - x_m^p|)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) = \|x_n^p - x_m^p\|_{1,\tau} = \|x^p E_n(x)E_m^\perp(x)\|_{1,\tau} \to 0$ as $n,m \to \infty$,

we get $\Phi(|x_n^p - x_m^p|) = e\Phi(|x_n^p - x_m^p|) \xrightarrow{t(\mathscr{B})} 0$. This means that $x^p \in L^1(M,\Phi)$ and $\Phi(x_n^p) \uparrow \widehat{\Phi}(x^p)$, i.e. $x \in L^p(Ms(\Phi_e), \Phi_e)$ $\|x\|_{p,\Phi_e} = \sup_{n\geq 1}(\Phi(x_n^p))^{\frac{1}{p}}$. Using the inequality $\nu(\Phi(x_n^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) = \tau(x_n^p) \leq \tau(x^p)$ we obtain that $\widehat{\Phi}(x^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \in L_1(\mathscr{B}, \nu)$ and

$$\nu(\widehat{\Phi}(x^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) = \sup_{n\geq 1} \tau(x_n^p) = \tau(x^p),$$

i.e. $\|x\|_{p,\tau} = \nu(\widehat{\Phi}(x^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1})$.

Since $\{x_\alpha\} \subset L^p(M,\Phi)$, we have that $x_\alpha s(\Phi_e) \in L^p(Ms(\Phi_e), \Phi_e)$, moreover $x_\alpha s(\Phi_e) \downarrow 0$. Let us show that $x = x_{\alpha_0} s(\Phi_e) \in L^p(Ms(\Phi_e), \tau)$. As above, we consider $x_n = E_n(x)x$. Since

$$0 \leq \Phi(x_n^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \uparrow \widehat{\Phi}(x^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \leq e,$$

we get $\tau(x_n^p) \leq \nu(e) < \infty$. Consequently, $x \in L^p(Ms(\Phi_e), \tau)$. The inequality $0 \leq x_\alpha \leq x_{\alpha_0}$, for $\alpha \geq \alpha_0$ implies $x_\alpha s(\Phi_e) \in L^p(Ms(\Phi_e), \tau)$ (see Theorem 3.5(*iii*)). Since $x_\alpha s(\Phi_e) \downarrow 0$ and the norm $\|\cdot\|_{p,\tau}$ is order continuous, we have $\|x_\alpha s(\Phi_e)\|_{p,\tau} \downarrow 0$, i.e. $\nu(e\widehat{\Phi}(x_\alpha^p)(\mathbf{1}_{\mathscr{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) \downarrow 0$. Hence, $e\widehat{\Phi}(x_\alpha)^p \downarrow 0$, which contradicts to the inequality $e\Phi(x_\alpha^p) \geq \varepsilon^p e$. $\square$

4. Duality for spaces $\mathbf{L^p(M, \Phi)}$

Let us start with the following property of L^p-spaces $L^p(M,\Phi)$.

Proposition 4.1. *If* $x \in L^p(M,\Phi)$, $y \in L^q(M,\Phi)$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, *then* $xy, yx \in L^1(M,\Phi)$ *and* $\widehat{\Phi}(xy) = \widehat{\Phi}(yx)$.

Proof. Without loss of generality, we can take $x \geq 0$, $y \geq 0$. It follows from Theorem 3.3 that $xy \in L^1(M,\Phi)$. Hence, $yx = y^*x^* = (xy)^* \in L^1(M,\Phi)$ and $\widehat{\Phi}(yx) = \widehat{\Phi}((xy)^*) = \overline{\widehat{\Phi}(xy)}$. Let $x_n = xE_n(x)$, $y_n = yE_n(y)$. Then $x_n, y_n \in M_+$ and $\|x - x_n\|_{p,\Phi} \xrightarrow{t(\mathscr{B})} 0$, $\|y - y_n\|_{q,\Phi} \xrightarrow{t(\mathscr{B})} 0$. Using the inequalities $|\widehat{\Phi}(xy) - \Phi(x_n y_n)| \leq |\widehat{\Phi}(xy) - \widehat{\Phi}(x_n y)| + |\widehat{\Phi}(x_n y) - \Phi(x_n y_n)| \leq \|x - x_n\|_{p,\Phi}\|y\|_{q,\Phi} + \|x_n\|_{p,\Phi}\|y - y_n\|_{q,\Phi}$, we obtain $\Phi(x_n y_n) \xrightarrow{t(\mathscr{B})} \widehat{\Phi}(xy)$. Since $\Phi(x_n y_n) = \Phi(\sqrt{x_n} y_n \sqrt{x_n}) \geq 0$ for all n, we get $\widehat{\Phi}(xy) \geq 0$. Therefore $\widehat{\Phi}(xy) = \overline{\widehat{\Phi}(xy)} = \widehat{\Phi}(yx)$. $\square$

Let $L^p(M,\Phi)^*$ be a BKS of all $S_h(\mathscr{B})$-bounded linear mappings from $L^p(M,\Phi)$ into $S(\mathscr{B})$, i.e. $S_h(\mathscr{B})$ is the dual space to the BKS $L^p(M,\Phi)$. It is clear that any $S_h(\mathscr{B})$-bounded linear operator T is a continuous mapping from $(L^p(M,\Phi), \|\cdot\|_{p,\Phi})$ into $(S(\mathscr{B}), t(\mathscr{B}))$, i.e., if x_α, $x \in L^p(M,\Phi)$, and $\|x_\alpha - x\|_{p,\Phi} \xrightarrow{t(\mathscr{B})} 0$, then $Tx_\alpha \xrightarrow{t(\mathscr{B})} Tx$.

Proposition 4.2. *(compare with* [1], *5.1.9). Let $T \in L^p(M,\Phi)^*$, $\psi : S(\mathscr{A}) \to S(\mathscr{B})$ be a $*$-isomorphism from Theorem 2.5(ii). Then $T(ax) = \psi(a)T(x)$ for all $a \in S(\mathscr{A})$, $x \in L^p(M,\Phi)$.*

Proof. By theorem 2.5(ii), for each $z \in P(\mathscr{A})$, $x \in L^p(M,\Phi)$ we have $\|zx\|_{p,\Phi} = \widehat{\Phi}(z|x|^p)^{\frac{1}{p}} = \psi(z)\widehat{\Phi}(|x|^p)^{\frac{1}{p}} = \psi(z)\|x\|_{p,\Phi}$. Since $T \in L^p(M,\Phi)^*$, $|Tx| \le c\|x\|_{p,\Phi}$ for some $c \in S_+(\mathscr{B})$ and all $x \in L^p(M,\Phi)$. Hence $|T(zx)| \le \psi(z)c\|x\|_{p,\Phi}$, i.e. the support $s(T(zx))$ is majorized by the projection $\psi(z)$. Multiplying the equality $T(x) = T(zx) + T((\mathbf{1}-z)x)$ by $\psi(z)$, we obtain

$$\psi(z)T(x) = \psi(z)T(zx) = T(zx).$$

If $a = \sum_{i=1}^n \lambda_i z_i$ is a simple element from $S(\mathscr{A})$, where $\lambda_i \in \mathbb{C}$, $z_i \in P(\mathscr{A})$, $i = 1,\dots,n$, then

$$T(ax) = \sum_{i=1}^n \lambda_i T(z_i x) = (\sum_{i=1}^n \lambda_i \psi(z_i))T(x) = \psi(a)T(x).$$

Let a be an arbitrary element from $S(\mathscr{A})$ and let $\{a_n\}$ be a sequence of simple elements from $S(\mathscr{A})$ such that $a_n \xrightarrow{t(\mathscr{A})} a$. Then $0 \le \psi(|a_n - a|) \xrightarrow{t(\mathscr{B})} 0$, $\psi(a_n) \xrightarrow{t(\mathscr{B})} \psi(a)$, and

$$\|a_n x - ax\|_{p,\Phi} = \widehat{\Phi}(|(a_n-a)x|^p)^{\frac{1}{p}} = \widehat{\Phi}(|a_n-a|^p|x|^p)^{\frac{1}{p}} = \psi(|a_n-a|)\|x\|_{p,\Phi} \xrightarrow{t(\mathscr{B})} 0.$$

Since T is continuous, $\psi(a_n)T(x) = T(a_n x) \xrightarrow{t(\mathscr{B})} T(ax)$. Due to the convergence $\psi(a_n)T(x) \xrightarrow{t(\mathscr{B})} \psi(a)T(x)$, the proof is complete. $\square$

Now we pass to description of the $S_h(\mathscr{B})$-dual space $L^p(M,\Phi)^*$.

Theorem 4.3. *Let Φ be an $S(\mathscr{B})$-valued Maharam trace on the von Neumann algebra M, $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $y \in L^q(M,\Phi)$, then the linear mapping $T_y(x) = \widehat{\Phi}(xy)$, $x \in L^p(M,\Phi)$, is $S(\mathscr{B})$-bounded and $\|T_y\| = \|y\|_{q,\Phi}$.

(ii) If $T \in L^p(M,\Phi)^*$, then there exists a unique $y \in L^q(M,\Phi)$ such that $T = T_y$.

Proof. (i) By the Hölder inequality (theorem 3.3), $xy \in L^1(M,\Phi)$ for all $x \in L^p(M,\Phi)$ and $|T_y(x)| = |\widehat{\Phi}(xy)| \le \|y\|_{q,\Phi}\|x\|_{p,\Phi}$. Hence, T_y is $S_h(\mathscr{B})$-bounded linear mapping from $L^p(M,\Phi)$ into $S(\mathscr{B})$. Due to Proposition 4.1 and Theorem 3.4 we have

$$\|T_y\| = \sup\{|\widehat{\Phi}(yx)| : x \in L^p(M,\Phi),\ \|x\|_{p,\Phi} \le \mathbf{1}_{\mathscr{B}}\} = \|y\|_{q,\Phi}.$$

(ii) Since $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathscr{B}}$, we can define the element $b = (\Phi(\mathbf{1}))^{-1} \in S_+(\mathscr{A})$. If $\Phi_1(x) = b\Phi(x),\ x \in M$, then $L^p(M, \Phi_1) = L^p(M, \Phi)$ and $\|x\|_{p,\Phi_1} = b^{\frac{1}{p}}\|x\|_{p,\Phi}$ for all $x \in L^p(M, \Phi)$. Therefore, one can take $\Phi(\mathbf{1}) = \mathbf{1}_{\mathscr{B}}$.

Let $T \in L^p(M, \Phi)^*$. We choose $a \in S_+(\mathscr{B})$ with $a\|T\| = s(\|T\|)$. Set $T_1(x) = aT(x),\ x \in L^p(M, \Phi)$. It is clear that $T_1 \in L^p(M, \Phi)^*$ and $\|T_1\| = a\|T\| = s(\|T\|) \leq \mathbf{1}_{\mathscr{B}}$. If we show that there exists $y_1 \in L^q(M, \Phi)$ such that $T_1 x = \Phi(xy_1)$, then by virtue of Proposition 4.2, $Tx = \|T\|T_1(xy_1) = T(x(\psi^{-1}(\|T\|)y_1)) = T(xy)$ where $y = \psi^{-1}(\|T\|)y_1 \in L^q(M, \Phi)$. Thus, one can also take that $\|T\| \leq \mathbf{1}_{\mathscr{B}}$.

At first, we assume that the algebra $\mathscr{B}$ is σ-finite. Let ν be a faithful normal finite numerical trace on $\mathscr{B}$. Since $|\Phi(x)| \leq \|x\|_M \Phi(\mathbf{1}) \leq \|x\|_M \mathbf{1}_{\mathscr{B}},\ x \in M$, we get $\Phi(x) \in L^1(\mathscr{B}, \nu)$. Consider on M the faithful normal finite trace $\tau(x) = \nu(\Phi(x)),\ x \in M$. Using the same trick as in the proof of Proposition 3.7, we can show that $L^p(M, \tau) \subset L^p(M, \Phi)$ and $\tau(|x|^p) = \|x\|_{p,\tau}^p = \nu(\widehat{\Phi}(|x|^p))$ for all $x \in L^p(M, \tau)$. Since $|T(x)| \leq \|x\|_{p,\Phi} = (\widehat{\Phi}(|x|^p))^{\frac{1}{p}}$, we have $T(x) \in L^1(\mathscr{B}, \nu)$ for all $x \in L^p(M, \tau)$.

We define on $L^p(M, \tau)$ the linear $\mathbb{C}$-valued functional $f(x) = \nu(Tx),\ x \in L^p(M, \tau)$. Since $|f(x)| \leq \nu(|T(x)|) \leq \nu(\widehat{\Phi}(|x|^p)^{\frac{1}{p}}\mathbf{1}_{\mathscr{B}}) \leq (\nu(\widehat{\Phi}(|x|^p)))^{\frac{1}{p}}(\nu(\mathbf{1}_{\mathscr{B}}))^{\frac{1}{q}} = (\nu(\mathbf{1}_{\mathscr{B}}))^{\frac{1}{q}}\|x\|_{p,\tau}$ for all $x \in L^p(M, \tau)$, we have that f is a bounded linear functional on $(L^p(M, \tau), \|\cdot\|_{p,\tau})$. Hence there exists an operator $y \in L^q(M, \tau) \subset L^q(M, \Phi)$ such that $f(x) = \tau(xy)$ for all $x \in L^p(M, \tau)$ [11]. We claim that $\tau(xy) = \nu(\widehat{\Phi}(xy))$ for all $x \in L^p(M, \tau)$. Let us remind that $\tau(|z|^p) = \nu(\widehat{\Phi}(|z|^p))$ for all $z \in L^p(M, \tau)$. If $z \in L^1_+(M, \tau)$, then $z^{\frac{1}{p}} \in L^p_+(M, \tau)$, and therefore $\tau(z) = \nu(\widehat{\Phi}(z)$. Hence, $\tau(z) = \nu(\widehat{\Phi}(z))$ for all $z \in L^1(M, \tau)$, in particular, $\tau(xy) = \nu(\widehat{\Phi}(xy))$ where $x \in L^p(M, \tau)$. Thus, $\nu(T(x)) = f(x) = \tau(xy) = \nu(\widehat{\Phi}(xy))$ for all $x \in L^p(M, \tau)$.

Let $T(x) - \widehat{\Phi}(xy) = v|T(x) - \widehat{\Phi}(xy)|$ be the polar decomposition of the element $(T(x) - \widehat{\Phi}(xy)) \in S(\mathscr{B})$ and take $a = \psi^{-1}(v^*)$. Since

$$0 = \nu(T(ax) - \widehat{\Phi}(axy)) = \nu(v^*(T(x) - \widehat{\Phi}(xy))) = \nu(|T(x) - \widehat{\Phi}(xy)|),$$

we have $T(x) = \widehat{\Phi}(xy)$ for all $x \in L^p(M, \tau)$.

Let $x \in L^p_+(M, \Phi), x_n = xE_n(x)$. Then $\|x_n - x\|_{p,\Phi} \xrightarrow{t(\mathscr{B})} 0$ and therefore $T(x_n) \xrightarrow{t(\mathscr{B})} T(x)$ and $|\widehat{\Phi}(x_n y) - \widehat{\Phi}(xy)| \leq \|x_n - x\|_{p,\Phi}\|y\|_{q,\Phi} \xrightarrow{t(\mathscr{B})} 0$. Since $T(x_n) = \widehat{\Phi}(x_n y)$, $T(x) = \widehat{\Phi}(xy)$, i.e. $T = T_y$.

If z is another element from $L^q(M, \Phi)$ with $T(x) = \widehat{\Phi}(xz), x \in L^p(M, \Phi)$, then $\widehat{\Phi}(x(y - z)) = 0$ for all $x \in L^p(M, \Phi)$. Taking $x = u^*$ where u is the unitary

operator from the polar decomposition $y-z = u|y-z|$, we obtain $\widehat{\Phi}(|y-z|) = 0$, i.e. $y = z$.

Now let $\mathscr{B}$ be a general (not necessarily a σ-finite) von Neumann algebra. Let ν be a faithful normal semifinite numerical trace on $\mathscr{B}$, and let $\{e_i\}_{i\in I}$ be a family of nonzero mutually orthogonal projections in $\mathscr{B}$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathscr{B}}$ and $\nu(e_i) < \infty$ for all $i \in I$. It is clear that $\mathscr{B}e_i$ is a σ-finite algebra and $\Phi_{e_i}(x) = e_i\Phi(x)$ is $S(\mathscr{B}e_i)$-valued Maharam trace on $Ms(\Phi_{e_i})$. Since $T \in L^p(M,\Phi)^*$, $T_i(x) = e_iT(x)$ is $S_h(\mathscr{B}e_i)$-bounded linear mapping onto $L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$. By virtue of what we proved above, there exists the unique $y_i \in L^q(Ms(\Phi_{e_i}), \Phi_{e_i})$, such that

$$e_iT(xs(\Phi_{e_i})) = \widehat{\Phi_{e_i}}(xs(\Phi_{e_i})y_i) = e_i\widehat{\Phi}(xs(\Phi_{e_i})y_i)$$

for all $x \in L^p(M,\Phi)$, $i \in I$. Moreover, $\|y_i\|_{q,\Phi} = \|T_i\| = \|T\|e_i$. Since $\sup\limits_{i\in I} s(\Phi_{e_i}) = \mathbf{1}$, $\{s(\Phi_{e_i})\}_{i\in I} \subset P(Z(M)$ and $s(\Phi_{e_i})s(\Phi_{e_j}) = 0$ as $i \neq j$, there exists a unique $y \in S(M)$ such that $ys(\Phi_{e_i}) = y_i$. We have $e_i\widehat{\Phi}(|y|^q) = \widehat{\Phi}(|y_i|^q) = \|T\|^q e_i$ for all $i \in I$. Hence, $y \in L^q(M,\Phi)$ $\|y\|_{q,\Phi} = \|T\|$ (see Proposition 3.1). In addition

$$e_i\widehat{\Phi}(xy) = \widehat{\Phi_{e_i}}(xs(\Phi_{e_i})y_i) = e_iT(xs(\Phi_{e_i})) = e_iT(x),$$

for all $i \in I$, i.e. $T_y(x) = \widehat{\Phi}(xy) = T(x)$, $x \in L^p(M,\Phi)$. $\square$

Corollary 4.4. *The BKS $L^p(M,\Phi)^*$ is isometric to the space $(L^q(M,\Phi), \|\cdot\|_{q,\Phi})$.*

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